Convergence of Random Variable

- $Y_n \xrightarrow{p} Y$: $\forall \epsilon > 0,\ P(|Y_n - Y| > \epsilon) \to 0$
- $Y_n \xrightarrow{d} Y$: $P(Y_n \le y) = F_{Y_n}(y) \to F_Y(y) = P(Y \le y)$
- $Y_n \xrightarrow{a.s.} Y$ ($Y_n \xrightarrow{w.p.1} Y$): $P(\{\omega : Y_n(\omega) \to Y(\omega)\}) = 1$
- $Y_n \xrightarrow{L^p} Y$: $E[|Y_n - Y|^p] \to 0$

LLN and CLT

- $x_1, \cdots, x_N \overset{iid}{\sim} X$, $\bar{x}_n = \frac{1}{n}\sum_{i=1}^n x_i$, $E[\bar{x}_n] = E[X] = \mu$.
- LLN: (S) $\bar{x}_n \xrightarrow{a.s.} \mu$ (assume $\sigma^2 < +\infty$); (W) $\bar{x}_n \xrightarrow{p} \mu$
- CLT: $(\bar{x}_n - \mu)/(\sigma/\sqrt{n}) \xrightarrow{d} N(0,1)$ (assume $\mu, \sigma^2 < +\infty$)

DTMC

- **Def**: $P(X_{n+1} = j | X_n = i, X_{n-1} = x_{n-1}, \cdots) = P(i,j)$
- $P(X_{m+i} = x_{m+i}, i \in [n] \mid X_i = x_i, i \in [m]) = \prod_{i=1}^n P(x_{m+i-1}, x_{m+i})$; $P_n(x,y) = P(X_{m+n} = y \mid X_m = x)$ depends on x, y, n
- **C-K Eqn**: $P_{m+n}(i,j) = \sum_{k \in \mathcal{X}} P_n(i,k) P_m(k,j)$
- **Stationary**: $\pi P = \pi$
- **Irreducible**: $i \leftrightarrow j$ for all $i, j \in \mathcal{X}$ (*Uniqueness*)
- **Aperiodic**: period:=gcd$\{n : P_n(i,i) > 0\} = 1$ (*Converge*) Irreducible+aperiodic $i \Rightarrow$ aperiodic DTMC
- **Stopping time**: A stopping time T is a r.v. with values in $\mathbb{N} \cup \{+\infty\}$ s.t. the event $\{T = m\}$ is determined by X_{0-m}
- **Strong Markov Property**: Let T be a stopping time for the MC $\{X_n, n \ge 0\}$. Then the MC regenerates at time T, i.e., $P(X_{T+i}, i \in [n] \mid T = m, X_i = x_i, i \in [m]) = \prod_{i=1}^n P(x_{m+i-1}, x_{m+i})$ [The M property holds for r.v. T]
- **Passage Time**: $T_x = \inf\{n : X_n = x\}$
- **Recurrence and Transience**: Let $\{X_n, n \ge 0\}$ be a MC on a countable state space $\mathcal{X}$. A state x is
  - recurrent: $P^x(T_x < \infty) = 1$; transient: $P^x(T_x < \infty) < 1$
  - positive recurrent: $E^x[T_x] < +\infty$ (*No escape to* $+\infty$)
  - null current: $P^x(T_x < \infty) = 1$, $E^x[T_x] = +\infty$
- **Thm**: Consider a DTMC on a state space $\mathcal{X}$ and $x \in \mathcal{X}$.
  - x recurrent $\Rightarrow P^x(X_n = x \text{ for } \infty\ n's) = 1$
  - x transient$\Rightarrow P^x(X_n = x \text{ for } \infty\ n's) = 0$
  - x recurrent $\Leftrightarrow \sum_{n=1}^\infty P_n(x,x) = \infty$
- **Thm**: Let $\{X_n, n \ge 0\}$ be a pos. recurrent and irreducible MC. For $x \in \mathcal{X}$, let N_n^x be the number of visits to state x up to time n. Then there exists a unique stationary distribution π, and for any $x \in \mathcal{X}$ and initial $y \in \mathcal{X}$,
$$\lim_{n\to\infty} N_n^x / n \xrightarrow{w.p.1} \pi(x);$$
if the MC is null recurrent, then there is no stationary distribution and for all $x \in \mathcal{X}$ and initial $y \in \mathcal{X}$,
$$\lim_{n\to\infty} N_n^x / n \xrightarrow{w.p.1} 0$$
- **Ergodic Thm**: Let $\{X_n, n \ge 0\}$ be a aperiodic, pos. recurrent and irreducible MC. Then there exists a unique stationary distribution π and
$$\lim_{n\to 0} P(X_n = x \mid X_0 = y) = \pi(x) \quad x, y \in \mathcal{X}.$$
The claim holds for any aperiodic, finite and irreducible MC, since finite + irreducible $\Rightarrow$ pos. recurrent.
- **Time Reversible**: The reverse process of a station. MC follows the same prob. law: $P^*(i,j) = P(i,j)$ for all $i, j \in \mathcal{X}$
- **Thm(Detailed Balance Eqn)**: For a pos. recurrent and irreducible MC, assume $\pi(j) \ge 0\ \forall j \in \mathcal{X}$, $e^\top \pi = 1$, and $\pi_i P(i,j) = \pi_j P(j,i)$ for all $i, j \in \mathcal{X}$. Then π is the station. distribution of this MC, and the process is time-reversible.

CTMC

Basic Structure

- **Def**: A CTMC defined on a countable state space $\mathcal{X}$ is a family $\{X(t), t \ge 0\}$ of $\mathcal{X}$-valued r.v.'s s.t. for all $t, s \ge 0$, $P(X(t+s) = x_{t+s} \mid X(u) = x_u, u \in [0,s]) = P(X(t+s) = x_{t+s} | X(s) = x_s)$
- **Holding Time**: τ_i is exponentially distributed with $v_i \ge 0$; instantaneous: $v_i = \infty$; absorbing: $v_i = 0$
- **Trans. Prob.**: $\sum_{j \ne i} P_{ij} = 1$ for all $i \in \mathcal{X}$
- **Embedded Chain**: The DTMC with the same trans. prob.
- **Trans. Rate**: $q_{ij} = v_i P_{ij}$, $\sum_{j \ne i} q_{ij} = \sum_{j \ne i} v_i p_{ij} = v_i$
- **C-K Eqn**: $P_{ij}(t+h) = \sum_{k \in \mathcal{X}} P_{ik}(h) P_{kj}(t)$

Birth and Death Process

- $v_i = \lambda_i + \mu_i$, $P_{i,i+1} = \lambda_i / (\lambda_i + \mu_1) = 1 - P_{i,i-1}$
- **Limit**: $P_n = \frac{\lambda_{n-1}}{\mu_n} P_{n-1}, P_0 = (1 + \sum_{n=1}^\infty \frac{\lambda_{n-1} \cdots \lambda_0}{\mu_n \cdots \mu_1})^{-1}$
- **Lin. Grow. with Immig.** $\lambda_n = n\lambda + \theta$, $\mu_n = n\mu$
- **Yule** ($X(0) = 1$): $\lambda_n = n\lambda$, $P(\sum_{i=1}^j T_i \le t) = (1 - e^{-\lambda t})^j$, $P_{1j}(t) = e^{-\lambda t}(1 - e^{-\lambda t})^{j-1}$, $p_{ij}(t) = C_{j-1}^{i-1} e^{-\lambda t i}(1 - e^{\lambda t})^{j-i}$
- **Epidemic with prob.** $\alpha h + o(h)$: $\lambda_n = n(m-n)\alpha$, $n \le m$-1

The Kolmogorov Differential Equation

- $\lim_{t\to 0} \frac{1 - P_{ii}}{t} = v_i$; $\lim_{t \to 0} \frac{P_{ij}}{t} = q_{ij} \quad \forall j \ne i$
- $P_{ij}(t+h) - P_{ij}(t) = \sum_{k \ne i} P_{ik}(h) P_{kj}(t) - [1 - P_{ii}(h)] P_{ij}(t)$ $h \to 0 \Rightarrow \mathbf{P'_{ij}(t) = \sum_{k \ne i} q_{ik} P_{kj}(t) - v_i P_{ij}(t)}$ (Backward)
- $P_{ij}(t+h) - P_{ij}(t) = \sum_{k \ne j} P_{ik}(t) P_{kj}(h) - [1 - P_{jj}(h)] P_{ij}(t)$ $h \to 0 \Rightarrow \mathbf{P'_{ij}(t) = \sum_{k \ne j} P_{ik}(t) q_{kj} - v_j P_{ij}(t)}$ (Forward)
- Forward Eqn regularity holds in B&D and finite-state

Limiting Probability

- **Prop**: A CTMC is irreducible iff so is its embedded chain
- **Prop**: A state x is recurrent/transient iff it is so in the embedded chain (but not for pos./null recurrent)
- **Prop**: For an irreducible CTMC, all states are transient, pos. recurrent or null recurrent, and always aperiodic.
- **Limiting distribution**: for any initial $i \in \mathcal{X}$, $P_j = \lim_{t\to\infty} \frac{1}{t} \int_0^t \mathbf{1}(\{X(s) = j | X(0) = i\}) ds\ w.p.1$
- **Thm**: Let $\{X(t), t \ge 0\}$ be an irreducible and pos.recurrent CTMC, then the limit distribution is unique:
$$P_j = E[\tau_j]/E^j[T_j] = 1/(v_j E^j[T_j]) > 0\ \forall j \in \mathcal{X}.$$
Moreover,
$$\lim_{t\to\infty} P_{ij}(t) \to P_j\ \forall i, j \in \mathcal{X}.$$
If the CTMC is null recurrent or transient, then $P_j = 0$ for all $j \in \mathcal{X}$.
- **Prop**: For an irreducible and pos.recurrent CTMC with limiting distribution P, $\sum_i P_i P_{ij} = P_j$ for all $j \in \mathcal{X}$. So P is also called **stationary**.
- **Thm**: If $\{X(t), t \ge 0\}$ is an irreducible CTMC with a pos.recurrent embedded chain, then the limiting probability P is unique and given by $P_j = (\pi_j / v_j)/(\sum_i \pi_i / v_i)$ for all $j \in \mathcal{X}$, where π is the station. distribution of the embedded chain. If $\sum_i \pi_i / v_i < +\infty$, then $P_j > 0$; o.w., $P_j = 0$.
- **Balance Eqn**: $v_j P_j = \sum_i P_i q_{ij}$, $\sum_j P_j = 1$
- **Thm**: An irreducible CTMC is pos.recurrent iff the balance equation has a prob. solution P, which is the limiting prob.
- **M/M/1**: $P_n = (\lambda/\mu)^n (1 - \lambda/\mu)$, $\lambda < \mu$

Time Reversibility

- **Prop**: The reverse process of a irreducible CTMC with limiting P is also a CTMC with the same P: $P(X(t-s) = j | X(t) = i, X(y), y > t) = P(X(t-s) = j | X(t) = j)$
- **Trans. prob. and rate** $P_{ij}^* = (\pi_j P_{ji})/\pi_i$, $q_{ij}^* = v_i P_{ij}^* = (v_i \pi_j P_{ji})/\pi_i$ $\pi_j / \pi_i = (v_j P_j)/(v_i P_i) \Rightarrow P_i q_{ij}^* = P_j q_{ji}$
- **Time Reversible**: $q_{ij}^* = q_{ij}$, or $P_i q_{ij} = P_j q_{ji}$ (Detailed BE)
- **Thm**: For an irreducible and pos. recurrent CTMC. Assume P is a prob. solution to $P_i q_{ij} = P_j q_{ji}$. Then P is the limiting prob. and the chain is time-reversible. (Note: DBE$\Rightarrow$ BE)
- **Prop**: An ergodic B&D is in steady state time reversible
- **Corr**: The output of M/M/s with $\lambda < s\mu$ is Poisson(λ)
- **Prop**: A t.r. chain with limiting P, that is truncated to the set $A \subseteq \mathcal{X}$ and remains irreducible is also t.r. and has limiting prob. $P_j^A = P_j / (\sum_{j \in A} P_j)$ for $j \in A$
- **M/M/1 with space** N: $P_j = (\lambda/\mu)^j / [\sum_{i=0}^N (\lambda/\mu)^i]$

Uniformalization

- $v_i = v$ for all $i \in \mathcal{X}$, then $\{N(t), t \ge 0\}$ is Poisson(v)
- $P_{ij}(t) = \sum_{n=0}^\infty P(X(t) = j | X(0) = i, N(t) = n) P(N(t) = n | X(0) = i) = \sum_{n=0}^\infty P_{ij}^n e^{-vt} \frac{(vt)^n}{n!}$
- Define CTMC with $P_{ij}^* = 1 - v_i / v$ ($i = j$) or $v_i P_{ij} / v$ ($i \ne j$)

Commonly Used Distribution

Distr.	*Mean*	*Var*	*pdf*	*cdf*
Uniform (a,b)	$\frac{1}{2}(a+b)$	$\frac{1}{12}(b-a)^2$	$\frac{1}{b-a}$	$\frac{x-a}{b-a}$
Binomial (n,p)	np	npq	$C_n^k p^k q^{n-k}$	
Geo (p)	$1/p$	$1-p/p$	$q^{n-1}p$	$1-q^n$
Poisson (λ)	λ	λ	$(\lambda^k e^{-\lambda})/k!$	
Exp (λ)	$1/\lambda$	$1/\lambda^2$	$\lambda e^{-\lambda x}$	$1-e^{-\lambda x}$